Online Appendix for

Government Procurement and Access to Credit: Firm Dynamics and Aggregate Implications

Julian di Giovanni[⋆] Manuel García-Santana[⋄] Priit Jeenas[▫]
Enrique Moral-Benito[∘] Josep Pijoan-Mas[▵]

June 10, 2022

Contents

A Details on the data **1**
- A.1 Public procurement in National Accounts 1
- A.2 Public procurement data 1
- A.3 Balance sheet data 3
- A.4 Credit data 3
- A.5 Types and size of procurement contracts 5
- A.6 Procurement firms across different industries 6
- A.7 Procurement vs. non-procurement firms 7
- A.8 Pre-trends for winners vs. the rest 8
- A.9 Robustness: Procurement and Credit across the Firm Size Distribution 9

B Details on the static production problem **1**
- B.1 Some preliminary results 2
- B.2 Firms without procurement 4
  - B.2.1 Unconstrained firms 4
  - B.2.2 Constrained firms 4
  - B.2.3 Binding constraints 8
- B.3 Firms with procurement 8
  - B.3.1 Unconstrained firms 8
  - B.3.2 Constrained firms. 9
- B.4 A procurement shock 13

B.4.1 Unconstrained firms 13
B.4.2 Constrained firms 14

C Details on some aggregates **18**
C.1 Equilibrium definition 18
C.2 Sectorial and aggregate TFP 18
C.3 Relative price of public sector good 19
C.4 Relative sectoral TFP 19
C.5 Relative sectoral $\overline{\text{MRPK}}$ 20

D Details on treatment effects in the benchmark calibration **21**

A Details on the data

A.1 Public procurement in National Accounts

According to the System of National Accounts (SNA), "Government consumption expenditures and gross investment", i.e., G, measures the fraction of GDP, or final expenditures, that is accounted for by the government sector. In that respect, the government is treated as a consumer/investor. In addition, the SNA treats the government as a producer that uses labor, capital, and intermediate goods to provide its own consumption and investment. The total value of this output, which equals G, is measured by the total cost incurred:

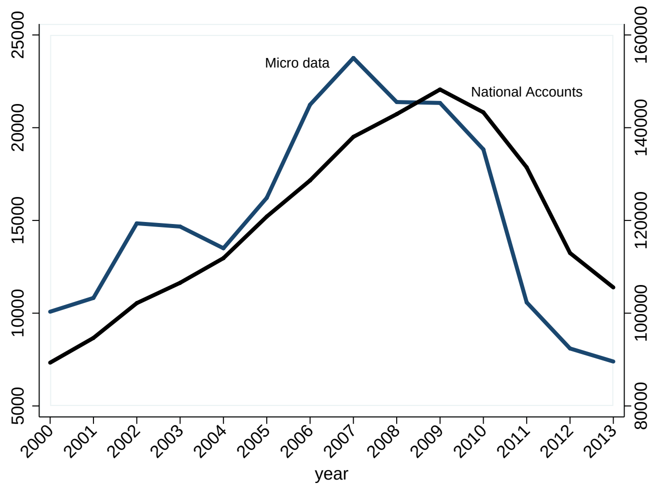
$$\begin{aligned} G &= \text{Gross Output of General Government} + \text{Gross Investment} \\ &= \text{Value Added} + \underbrace{\text{Intermediate goods and services} + \text{Structures} + \text{Equipment}}_{\text{public procurement}} + \text{IPP} \end{aligned}$$

Figure A.I shows the evolution of procurement value as measured with our micro data and compare it to the counterpart from national accounts. On average, our micro data accounts for around 13% of total government procurement as measured in Spanish national accounts. As apparent in the figure, our micro data reproduces well the cyclical aspect of pubic procurement expenditure, increasing during the boom and decreasing during the recession.

A.2 Public procurement data

Public procurement is defined in the *System of National Accounts (SNA)* as the sum of intermediate consumption (e.g., purchases of goods like medical consumables and services like accounting services), gross capital formation (e.g., building new roads), and social transfers in kind via market producers (e.g., medicines). Roughly speaking, one can think of public procurement as "government consumption expenditures and

Figure A.I. Evolution of Public Procurement in Spain, 2000-13



Notes: This figure shows the evolution of public procurement in Spain over 2000-13. The blue line ("Micro data", left y-axis) is computed by aggregating the individual projects scraped from the BOE, https://www.boe.es/. The black line ("National accounts", right y-axis) is measured from Spanish national accounts.

gross investment" (the G part of GDP) minus "compensation of employees" and "consumption of fixed capital." The size of public procurement varies across countries and over time. For the case of OECD countries, public procurement represented approximately 12% of GDP and 30% of G averaged over the 2007-2017 period.

Main sample of projects published in BOE. According to Spanish law, all procurement contracts above a certain threshold awarded by public institutions must be published in official bulletins.[1] If the contract is awarded by the central government, the information on this contract must be published in the *Agencia Estatal Boletín Oficial del Estado* (BOE), which is the official bulletin of the central government of Spain. In contrast, if the entity that awards the contract is a regional government or a municipality, the information about this contract can alternatively be published at their respective regional or local bulletin.

We construct a novel dataset on Spanish public procurement contracts by scraping the BOE website over the 2000-2013 period. Each contract provides considerable information on each awarded project. In particular, we collect information on the type of contract (kind of good or service provided), the institution awarding the contract, the initial bidding and final price of the contract, the type of procedure used to allocate the contract, and the firm(s) that won the contract. In total, we scraped more than 150,000 projects over 2000-2013, which we assign to the month

[1]The thresholds above which the contract must be advertised in official bulletins depend on the type of contract. In the case of supplies and services, for example, the threshold is 60,000 euros.

that the project was awarded. Of these, 130,633 projects have a value assigned to them that we were able to recover. The sum of all these projects totals around 220 billion euros. On average, our micro data account for around 13% of total public procurement as measured in Spanish National Accounts. Despite the level differences, our micro data are able to capture the overall evolution of public procurement over time, which increased from 9.9 to 13.8 percent between 2000 and 2009 and decreased from 13.8 to 10.0 percent between 2010 and 2013; see Figure A.I.

Small sample of projects with information on bidders. Although the BOE website provides detailed information about the characteristics of the contracts, it does not provide the identity of the firms that competed for the project but did not win. This is a limitation of our dataset because it does not allow us to construct a well-defined control group. To overcome this limitation, we construct a sample of procurement projects for which we have detailed information about the awarding process. Although we did not find any government agency that provided information about the awarding process during our main sample period (2000-2013), we could identify around 50 agencies that started providing detailed information about their projects starting in 2013. Putting all these agencies together, we were able to uncover the identity of the firms competing for the same projects as well as their final rankings for around 1,000 contracts over the 2013-2016 period.

A.3 Balance sheet data

We use the balance sheets and income statements of the quasi-universe of Spanish companies between 2000 and 2016, a dataset that is maintained by the Banco de España and taken from the Spanish Commercial Registry. For each firm and year, this dataset includes information on the firm's name, fiscal identifier, sector of activity (4-digit NACE Rev. 2 code), age, net operating revenue, material expenditures, number of employees, labor expenditures, total fixed assets, total assets, and net worth. The final sample covers a total of 1,801,955 firms with an average of 993,876 firms per year. This represents around 85-90% of the firms in the non-financial market economy for all size categories in terms of both turnover and number of employees. This database is used by García-Santana et al. (2020) among others and is described in detail by Almunia et al. (2018).

A.4 Credit data

The *Central de Información de Riesgos* (CIR) is maintained by the Banco de España in its role as primary banking supervisory agency, and contains detailed monthly information on all outstanding loans over 6,000 euros to non-financial firms granted by all banks operating in Spain since 1984. Given the low reporting threshold, virtually all firms with outstanding bank debt appear in the CIR. In addition to the total amount of credit, CIR also contains information on whether or not a non-personal collateral ("Garantía real") was posted for a particular loan. These collaterals include assets like real estate, land, machinery, securities, deposits, and merchandise (i.e., hard collateral).[2] With this information, we can hence assess whether a particular loan for a bank-firm pair was granted on the basis of tangible collateral. We use data from 2000 to 2016.

Loan applications. Besides the information on outstanding loans, we also have information about loan applications at the firm-bank level. The construction of this dataset is as follows. Spanish banks can request information about a firm whenever this firm "seriously" approaches them to obtain credit.[3] Because banks already have information about the firms with which they have a credit relationship, banks only request information on firms that have never received a loan from them or that ended the credit relationship before the current request. By matching the loan applications with the information on outstanding loans from CIR, we can infer whether the loan was granted or not.[4]

[2] See Ivashina et al. (Forthcoming) for more details.

[3] The Law stipulates that a bank can not request information about the firm without its consent, which indicates the seriousness of the approach

[4] Both the CIR and loan application data provide the identity, i.e., fiscal identifier, of the firm involved in every loan, allowing us to easily match the loan data with the balance sheet and income statements of firms.

A.5 Types and size of procurement contracts

Table A.I provides descriptive evidence for the pool of projects and years between 2000 and 2013. In particular, we report statistics on the number of projects and distribution of projects' values, separately for the five broad sector categories reported by the BOE: construction, consulting, services, supplies, and others.

Table A.I. Value of Procurement projects (budget value in millions of euro), pool of years 2000-13

Sector	mean	10th	25th	50th	75th	99th	obs.
Construction	5.28	0.13	0.23	0.74	4.00	70.84	22,549
Consulting	0.66	0.10	0.17	0.37	0.84	3.91	12,427
Services	1.22	0.11	0.20	0.42	1.05	13.47	44,581
Supplies	0.95	0.10	0.17	0.37	0.86	10.20	45,552
Others	1.99	0.09	0.15	0.35	0.99	38.18	5,524

Notes: This table presents summary statistics on the size of procurement projects in our sample as measured by the budget value. All the numbers have been computed after trimming the top 1% of the projects in terms of value, which mostly correspond to typos in the numbers, e.g., displaced comma, reported in BOE.

A.6 Procurement firms across different industries

In Table A.II, we present summary statistics for the top 20 NACE 2-digit industries in terms of the fraction of firms in that industry that sell to the government. In column (1), we show the share of firms active in procurement in that industry. In columns (2-5), we show the share of employment, sales, fixed assets and credit accounted for by procurement firms in that industry. We provide the numbers for the year 2006, but the ranking and shares are similar for the rest of the years.

Table A.II. Importance of procurement firms, 2006

Sector	Description	Firms (1)	Emp. (2)	Sales (3)	Assets (4)	Credit (5)
19	Manufacture of coke & refined petroleum prod.	0.150	0.332	0.315	0.310	0.243
21	Manufacturing of Pharmaceutical Products	0.149	0.240	0.225	0.231	0.288
42	Civil Engineering	0.093	0.260	0.324	0.366	0.386
80	Security and investigation activities	0.064	0.198	0.299	0.269	0.312
30	Manufacturing of Transport Equipment	0.052	0.176	0.177	0.205	0.180
94	Activities of membership organisations	0.051	0.069	0.127	0.037	0.018
36	Collection, purification and distribution of water	0.040	0.116	0.117	0.088	0.121
61	Telecommunications	0.038	0.217	0.192	0.189	0.207
51	Air transportation	0.033	0.054	0.049	0.078	0.142
81	Services of Buildings Maintenance	0.031	0.137	0.232	0.151	0.211
63	Information services	0.026	0.127	0.100	0.080	0.087
62	Programming, consultancy, other IT activities	0.025	0.151	0.193	0.157	0.214
26	Manufacturing of IT, electronic, & optical prod.	0.025	0.087	0.095	0.125	0.165
71	Technical services of architecture & engineering	0.024	0.152	0.159	0.084	0.103
2	Forestry and logging	0.019	0.069	0.068	0.033	0.080
6	Extraction of crude petroleum and natural gas	0.017	0.021	0.036	0.016	0.026
91	Libraries, archives, museums and cultural activities	0.016	0.061	0.051	0.021	0.017
29	Manufacture of motor vehicles and trailers	0.015	0.030	0.036	0.030	0.086
72	R&D activities	0.014	0.017	0.014	0.003	0.003
17	Paper industry	0.014	0.033	0.032	0.038	0.067

Notes: This table presents summary statistics on the number of firms, employment sales, and credit for the year 2006 at the 2-digit sectors. 'Firms' refers to the share of procurement firms, 'Emp.', 'Sales', 'Assets' , and 'Credit' are the share of employment, sales, assets and credit accounted for by procurement firms.

A.7 Procurement vs. non-procurement firms

Table A.III. Descriptive evidence from the final merged dataset, year 2006

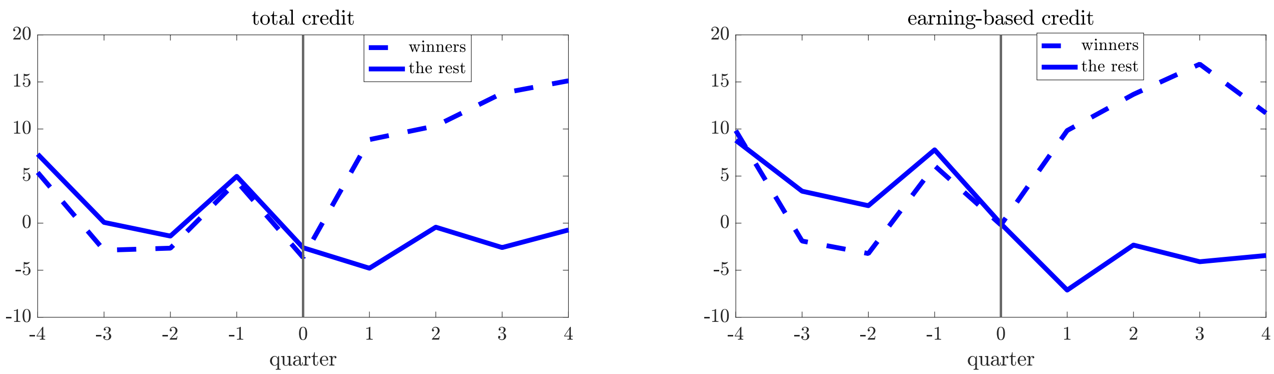
	mean		**25th pctile**		**50th pctile**		**75th pctile**	
	Proc	NoProc	Proc	NoProc	Proc	NoProc	Proc	NoProc
Age	20.42	10.95	12.00	5.00	17.00	10.00	24.00	15.00
Employment	73.56	12.75	16.00	3.00	45.00	6.00	155.0	12.00
Sales	8.96	1.19	1.14	0.10	4.22	0.28	16.89	0.86
Procurement/Sales	0.20	0.00	0.01	0.00	0.03	0.00	0.10	0.00
Fixed Assets	3.80	0.85	0.21	0.03	0.82	0.14	3.58	0.50
Net worth	3.92	0.43	0.36	0.01	1.27	0.07	6.12	0.30
Credit	2.51	0.57	0.11	0.03	0.48	0.08	2.32	0.30
Coll. Credit (share)	0.14	0.29	0.00	0.00	0.00	0.00	0.14	0.74

Notes: This table presents summary statistics from our merged dataset for the year 2006, separately for firms with at least one procurement contract ($n = 2,411$) vs. the rest of the firms ($n = 406,261$). The variable *Employment* measures the number of full-time workers employed by the firm; the variable *Sales* is just firm's revenue measured in millions of euro; *Procurement/Sales* measures the value of all the procurement projects awarded to a firm in a given year divided by total revenue in that year; *Assets* measures the value of fixed assets; *net worth* measures total assets minus total debt; *Credit* measures the value of all firm's outstanding loans in millions of euro; *Coll. Credit (share)* is the share of *Credit* collateralized against firm's assets; *Def. Credit (share)* is the share of defaulted credit over total *Credit*; *age* measures the age of the firm. We winsorize the 1% tails of all variables to make numbers result to outliers.

A.8 Pre-trends for winners vs. the rest

Graphically, the right panel in Figure A.II shows the average growth of credit without collateral of firms that win a procurement project in quarter 0 before and after winning the project, and compares it to the rest of firms. Again, there is a similar evolution of credit growth before procurement (parallel trends) and a clear (and persistent) divergence after that.

Figure A.II. Credit Growth: bidders sample



Notes: These graphs plot the evolution of the average change in credit for winning vs. non-winning firms, before and after the quarter in which the auction takes place (Quarter=0). The left panel is for all credit. The right panel is for non-collateral credit only.

A.9 Robustness: Procurement and Credit across the Firm Size Distribution

Table A.IV. Credit Growth and Procurement: Estimation across the Firm Distribution

	Assets (1)	Employment (2)	Net Worth (3)	Age (4)
Quartile 1	0.057	0.028	0.040	0.029
	(0.009)	(0.008)	(0.008)	(0.008)
Quartile 2	0.040	0.065	0.054	0.039
	(0.009)	(0.009)	(0.009)	(0.009)
Quartile 3	0.051	0.058	0.049	0.068
	(0.009)	(0.009)	(0.009)	(0.009)
Quartile 4	0.070	0.070	0.071	0.065
	(0.009)	(0.009)	(0.009)	(0.009)

Notes: This table presents results from estimating the relationship between total credit growth and procurement participation (PROC) by regression (1) with firms obtaining at least one procurement project over 2000-13. Only the coefficient of the PROC dummy is present for each quartiels of the firm distribution based on: (1) total assets, (2) employment, (3) net worth, and (4) age. All regressions use quarterly data. Standard errors are clustered at the firm level and all coefficients are significant at the 1% level.

B Details on the static production problem

In this Appendix we characterize the solution of the static production problem. First, in Section B.1 we derive the results that serve to restrict the parameters ϕ_p and ϕ_g such that the problem is well-behaved. Then, in Section B.2 we characterize analytically the solution to the production problem for firms without procurement ($d = 0$), which is useful to understand the interaction of asset based and earnings based financial constraints. Next, in Section B.3 we characterize analytically some of the solutions to the production problem for firms with procurement ($d = 1$) for the case $\sigma_p = \sigma_g$. Finally, in Section B.4 we show analytically the effect of a procurement shock for the case $\sigma_p = \sigma_g$, that is, the differences in allocations and profits between a firm with $(s, a, d = 1)$ and a firm with $(s, a, d = 0)$.

Before going to all these results, we start the Appendix by rewriting the FOC of the static production problem as follows. First, note that because the FOC for u, equation (11), states that the marginal revenue per unit of output sold —including its value as collateral— has to be equalized across the two sectors, and using the fact that $\frac{\partial p_p y_p}{\partial k} / \frac{\partial p_p y_p}{\partial u} = u/k$ and $\frac{\partial p_g y_g}{\partial k} / \frac{\partial p_g y_g}{\partial (1-u)} = (1-u)/k$ we can write the FOC for k, equation (12), as,

$$\frac{\partial p_p y_p}{\partial k} \frac{1}{u} = \frac{\partial p_p y_p}{\partial k_p} = \frac{r + \delta + \lambda}{1 + \lambda \phi_p} \tag{B.1}$$

or as

$$\frac{\partial p_g y_g}{\partial k} \frac{1}{1-u} = \frac{\partial p_g y_g}{\partial k_g} = \frac{r + \delta + \lambda}{1 + \lambda \phi_g} \tag{B.2}$$

or combining them both,

$$\frac{\partial \left[p_p y_p + p_g y_g \right]}{\partial k} = u \left(\frac{r + \delta + \lambda}{1 + \lambda \phi_p} \right) + (1-u) \left(\frac{r + \delta + \lambda}{1 + \lambda \phi_g} \right)$$

That is, the revenue marginal product of capital in each sector is equal to the capital cost of each sector and the revenue marginal product of capital for the whole firm is a weighted average of the capital costs in the two sectors, with the weights given but the cost shares of each sector.

It will be useful later on to use the actual revenue functions and substitute in equations (B.1) and (B.2) to obtain,

$$\left(\frac{\sigma_p - 1}{\sigma_p} \right) \frac{p_p y_p}{k} \frac{1}{u} = \frac{r + \delta + \lambda}{1 + \phi_p \lambda} \tag{B.3}$$

$$\left(\frac{\sigma_g - 1}{\sigma_g} \right) \frac{p_g y_g}{k} \frac{1}{1-u} = \frac{r + \delta + \lambda}{1 + \phi_g \lambda} \tag{B.4}$$

and using the production function one can write them as

$$\left(\frac{\sigma_p-1}{\sigma_p}\right)p_p\, s=\frac{r+\delta+\lambda}{1+\lambda\phi_p} \tag{B.5}$$

$$\left(\frac{\sigma_g-1}{\sigma_g}\right)p_g\, s=\frac{r+\delta+\lambda}{1+\lambda\phi_g} \tag{B.6}$$

Finally, dividing these two equations we get an expression for the optimal relative prices,

$$\frac{p_p}{p_g}=\frac{1+\lambda\phi_g}{1+\lambda\phi_p}\frac{(\sigma_g-1)/\sigma_g}{(\sigma_p-1)/\sigma_p} \tag{B.7}$$

Note that whenever $\sigma_p=\sigma_g$, $p_g/p_p=1$ for firms without binding financial frictions ($\lambda=0$). For firms with binding financial frictions ($\lambda>0$) $p_g/p_p<1$ ($p_g/p_p>1$) whenever $\phi_g>\phi_p$ ($\phi_g<\phi_p$) because production is shifted towards the sector that provides better collateral, and $p_g/p_p=1$ whenever $\phi_g=\phi_p$.

B.1 Some preliminary results

Lemma 1 *The terms* $\frac{r+\delta+\lambda}{1+\lambda\phi_p}$ *and* $\frac{r+\delta+\lambda}{1+\lambda\phi_g}$ *describing the cost of capital for the production of the private sector and the public sector goods respectively, are (a) strictly below* $1/\phi_p$ *and* $1/\phi_g$ *respectively, (b) increasing in* λ*, and (c) strictly above* $r+\delta$ *when* $\lambda>0$*, if and only if* $\phi_p<(\delta+r)^{-1}$ *and* $\phi_g<(\delta+r)^{-1}$ *respectively.*

Proof: Part (a) is straightforward:

$$\frac{r+\delta+\lambda}{1+\lambda\phi_p}<\frac{1}{\phi_p}\Leftrightarrow\phi_p(r+\delta+\lambda)<(1+\lambda\phi_p)\Leftrightarrow\phi_p(r+\delta)<1\Leftrightarrow\phi_p<(r+\delta)^{-1}$$

For part (b) note that

$$\frac{d}{d\lambda}\left(\frac{r+\delta+\lambda}{1+\lambda\phi_p}\right)\propto(1+\lambda\phi_p)-\phi_p(r+\delta+\lambda)>0\Leftrightarrow\phi_p(r+\delta)<1\Leftrightarrow\phi_p<(r+\delta)^{-1}$$

Finally, part (c) is proved by noting that $\frac{r+\delta+\lambda}{1+\lambda\phi_p}$ equals $r+\delta$ whenever $\lambda=0$ and its derivative w.r.t. λ is positive, see part (b). The same arguments apply for $\frac{r+\delta+\lambda}{1+\lambda\phi_g}$. ■

Proposition 1 *Holding* s *constant, more constrained firms sell less to the private sector, sell less to the public sector, and demand less capital if both* $\phi_p<(\delta+r)^{-1}$ *and* $\phi_g<(\delta+r)^{-1}$.

Proof: Let's combine the FOC (B.3) with the demand equation in (4) to produce the expression,

$$y_p=\left(\frac{\sigma_p-1}{\sigma_p}B_p s\frac{1+\lambda\phi_p}{r+\delta+\lambda}\right)^{\sigma_p}$$

Then, by virtue of Lemma 1 y_p falls with λ whenever $\phi_p < (\delta + r)^{-1}$. The case for y_g is analogous. Finally, note that total output is split between private sector and public sector sales, that is, $y_p + y_g = f(s, k) = sk$, so the derivative of capital with respect to λ is just,

$$\frac{dk}{d\lambda} = \frac{1}{s}\left(\frac{dy_p}{d\lambda} + \frac{dy_g}{d\lambda}\right)$$

which is negative given the previous results in this Proposition. ■

Lemma 2 *The optimal unconstrained capital for the private and the public sector respectively cannot be self-financed through its own revenues if and only if* $\phi_p \frac{\sigma_p}{\sigma_p - 1}(r + \delta) < 1$ *and* $\phi_g \frac{\sigma_g}{\sigma_g - 1}(r + \delta) < 1$ *respectively.*

Proof: The optimal unconstrained solution for the private sector capital is given by equation (B.3) when $\lambda = 0$, which implies $\frac{p_p y_p}{k}\frac{1}{u} = \frac{\sigma_p}{\sigma_p - 1}(r + \delta)$. When $\phi_p \frac{\sigma_p}{\sigma_p - 1}(r + \delta) < 1 \Leftrightarrow \frac{\sigma_p}{\sigma_p - 1}(r + \delta) < \phi_p^{-1}$ this leads to $\frac{p_p y_p}{k}\frac{1}{u} < \phi_p^{-1} \Leftrightarrow \phi_p p_p y_p < uk$, that is, the optimal unconstrained capital for the private sector, uk, cannot be self-financed through its own revenues. The proof for the public sector capital is analogous by use of the FOC (B.4) ■

Proposition 2 *Entrepreneurs with zero net worth are financially constrained if both* $\phi_p \frac{\sigma_p}{\sigma_p - 1}(r + \delta) < 1$ *and* $\phi_g \frac{\sigma_g}{\sigma_g - 1}(r + \delta) < 1$.

Proof: Note that if both $\phi_p \frac{\sigma_p}{\sigma_p - 1}(r + \delta) < 1$ and $\phi_g \frac{\sigma_g}{\sigma_g - 1}(r + \delta) < 1$, then following Lemma 2 both $\phi_p p_p y_p < uk$ and $\phi_g p_g y_g < (1 - u)k$. Adding them up leads to $\phi_p p_p y_p + \phi_g p_g y_g < k$, which implies that the capital of the unconstrained solution cannot be financed through revenue based constraints and hence entrepreneurs with zero net worth are constrained. ■

Lemma 3 *The term* $\phi_p \frac{\partial p_p y_p}{\partial k} + \phi_g \frac{\partial p_g y_g}{\partial k}$ *describing the share of capital that can be self-financed through revenues is positive and strictly smaller than one for constrained firms.*

Proof: That this term is positive is straightforward. To show that it is lower than one, note that for constrained firms the borrowing constrain in (9) holds with equality. Hence, for $a \geq 0$ it must be that $k \geq \phi_p p_p y_p + \phi_g p_g y_g$ or $\phi_p \frac{p_p y_p}{k} + \phi_g \frac{p_g y_g}{k} \leq 1$ (with strict equality for $a = 0$). Given our revenue function, the marginal products are proportional to the average products $\frac{\partial p_p y_p}{\partial k} = \left(\frac{\sigma_p - 1}{\sigma_p}\right)\frac{y_p p_p}{k}$ and $\frac{\partial p_g y_g}{\partial k} = \left(\frac{\sigma_g - 1}{\sigma_g}\right)\frac{y_g p_g}{k}$, so we can rewrite

$$\phi_p \frac{\partial p_p y_p}{\partial k} + \phi_g \frac{\partial p_g y_g}{\partial k} = \phi_p \left(\frac{\sigma_p - 1}{\sigma_p}\right)\frac{p_p y_p}{k} + \phi_g \left(\frac{\sigma_g - 1}{\sigma_g}\right)\frac{p_g y_g}{k}$$

Note that $\sigma_p > 1$ and $\sigma_g > 1$ implies that $\frac{\sigma_p - 1}{\sigma_p} < 1$ and $\frac{\sigma_g - 1}{\sigma_g} < 1$ (the marginal products are below the average products), and hence it is the case that $\phi_p \frac{\partial p_p y_p}{\partial k} + \phi_g \frac{\partial p_g y_g}{\partial k} < \phi_p \frac{p_p y_p}{k} + \phi_g \frac{p_g y_g}{k} \leq 1$ ■

Lemma 4 *The term $\phi_p \frac{\partial p_p y_p}{\partial u} + \phi_g \frac{\partial p_g y_g}{\partial u}$ describing the increase in credit that can be achieved by reallocation output to the private sector has the sign of $(\phi_p - \phi_g)$ for constrained firms.*

Proof: Using equation (11) we can write:

$$\phi_p \frac{\partial p_p y_p}{\partial u} + \phi_g \frac{\partial p_g y_g}{\partial u} = \frac{\partial p_p y_p}{\partial u} \left[\phi_p - \phi_g \frac{1 + \lambda \phi_p}{1 + \lambda \phi_g} \right] = \frac{\partial p_p y_p}{\partial u} \phi_g \left[\frac{\phi_p}{\phi_g} - \frac{\lambda^{-1} + \phi_p}{\lambda^{-1} + \phi_g} \right]$$

Note that with $\phi_p > \phi_g$ ($\phi_p < \phi_g$), this expression is positive (negative) when λ tends to zero, it decreases (increases) monotonically with λ, and tends to zero when λ tends to infinity. ■

B.2 Firms without procurement

We start analyzing the production problem for firms without procurement, that is, firms with $d = 0$.

B.2.1 Unconstrained firms

With $\lambda = 0$ the FOC for k in (12) becomes,

$$\frac{\partial p_p y_p}{\partial k} = r + \delta$$

which states that firms must equalize the marginal revenue product of capital to the cost of capital. This equation defines the optimal demand of capital $k^* (s, a, 0)$ for every entrepreneur of type $(s, a, d = 0)$. In particular, one gets $\frac{\sigma - 1}{\sigma} \frac{p_p y_p}{k} = r + \delta$ and substituting for the revenue function yields the optimal demand for capital

$$k^* (s, a, 0) = \left[\left(\frac{\sigma_p - 1}{\sigma_p} \right) \frac{B_p}{r + \delta} \right]^{\sigma} s^{\sigma - 1} \tag{B.8}$$

Next, note that profits are given by $\pi = p_p y_p - (r + \delta) k$, which given the optimal choice of capital can be written as $\pi = \frac{1}{\sigma_p} p_p y_p$ or $\pi = \frac{1}{\sigma_p - 1} (r + \delta) k$. Substituting optimal capital demand to the revenue function gives $p_p y_p = B_p \left[\left(\frac{\sigma_p - 1}{\sigma_p} \right) \frac{B_p}{r + \delta} \right]^{\sigma_p - 1} s^{\sigma_p - 1}$, which can be substituted back to the profit function to obtain:

$$\pi^* (s, a, 0) = \frac{1}{\sigma_p} \left[\left(\frac{\sigma_p - 1}{\sigma_p} \right) \frac{1}{r + \delta} \right]^{\sigma_p - 1} B_p^{\sigma} s^{\sigma_p - 1} \tag{B.9}$$

Hence, capital demand and profits increase monotonically with the shock s and are independent from net worth a.

B.2.2 Constrained firms

If the firm is constrained, then $\lambda > 0$ and the FOC of the problem are:

$$(1+\lambda\phi_p)\frac{\partial p_p y_p}{\partial k} = r + \delta + \lambda \tag{B.10}$$

$$k = \phi_a a + \phi_p p_p y_p \tag{B.11}$$

which determine k and λ. In particular, the borrowing constraint, equation (B.11), defines the capital demand $k(s,a,0)$, the FOC, equation (B.10), delivers the shadow value of the constrain $\lambda(s,a,0)$, and the objective function delivers the profit function $\pi(s,a,0)$. The next propositions chacarcterize the derivatives of these three functions with respect to the state variables a and s.

Let's start by totally differentiating equation (B.11) in turns with respect to a and s to obtain,

$$\frac{\partial k(s,a,0)}{\partial a} = \phi_a \left(1-\phi_p \frac{\partial p_p y_p}{\partial k}\right)^{-1} \tag{B.12}$$

$$\frac{\partial k(s,a,0)}{\partial s} = \phi_p \frac{\partial p_p y_p}{\partial s}\left(1-\phi_p \frac{\partial p_p y_p}{\partial k}\right)^{-1} \tag{B.13}$$

With $\phi_p = 0$ we are in the case without earnings-based collateral constraints and these derivatives are just equal to ϕ_a and 0 respectively: higher net worth allows to operate with more capital but higher productivity does not. With $\phi_p > 0$ both derivatives are positive, that is, constrained firms with more net worth or higher productivity operate with more capital. Indeed, in this case $\frac{\partial k(s,a,0)}{\partial a} > \phi_a$ because an increase in net worth has a multiplier effect through the increase in revenues and the easing of the earnings-based financial constraint (see Lemma 3). This is stated in the next proposition:

Proposition 3 *The derivative of $k(s,a,0)$ with respect to a is positive, while the derivative of $k(s,a,0)$ with respect to s is positive as long as $\phi_p > 0$ (and zero othwewise).*

Proof: The derivatives of $k(s,a,0)$ with respect to a and s are given by equation (B.12) and (B.13). $\phi_a \geq 1$ and Lemma 3 states that $\phi_p \frac{\partial p_p y_p}{\partial k} < 1$, so the derivative with respect to a is strictly positive. For the derivative with respect to s, note additionally that $\frac{\partial p_p y_p}{\partial s} > 0$. Hence, this derivative is strictly positive (zero) if $\phi_p > 0$ ($\phi_p = 0$). ■

Note also that the derivatives of capital with respect to a and s are higher for more constrained firms (higher λ) because the multiplier effect of the earnings-based constraints is larger for firms with higher marginal product of capital, that is, the increase in capital demand with net worth a or productivity s is larger for more financially constrained firms. This is stated in the next corollary:

Corollary 1 *The derivatives of* $k(s,a,0)$ *with respect to* a *and with respect to* s *increase with* λ

Proof: The derivatives are characterized by equations (B.12) and (B.13). Using the FOC (B.10) and the fact that $\frac{\partial p_p y_p}{\partial s} = \frac{k}{s}\frac{\partial p_p y_p}{\partial k}$ we can further rewrite them as

$$\frac{\partial k(s,a,0)}{\partial a} = \phi_a \left(1 - \phi_p \frac{r+\delta+\lambda}{1+\lambda\phi_p}\right)^{-1} \quad \text{(B.14)}$$

$$\frac{\partial k(s,a,0)}{\partial s} = \phi_p \frac{k}{s}\frac{r+\delta+\lambda}{1+\lambda\phi_p}\left(1 - \phi_p \frac{r+\delta+\lambda}{1+\lambda\phi_p}\right)^{-1} \quad \text{(B.15)}$$

To prove this corollary it is enough to show that the term $(r+\delta+\lambda)/(1+\lambda\phi_p)$ in equations (B.32) and (B.33) increases with λ, which is proved in Lemma 1. ■

Next, equation (B.10) allows to recover $\lambda(s,a,0)$. It can be shown that $\lambda(s,a,0)$ declines with a —wealthier entrepreneurs can finance larger amounts of capital and are hence less constrained— and increases with s —s increases optimal capital by more than it increases the amount of capital that can be self-financed through revenues. This is stated formally in Proposition 4.

Proposition 4 *The derivative of* $\lambda(s,a,0)$ *with respect to* a *is always negative, while the derivative of* $\lambda(s,a,0)$ *with respect to* s *is always positive as long as* $a>0$ *(and zero otherwise).*

Proof: Equation (B.10) can be rewritten as

$$\frac{\partial p_p y_p}{\partial k} = \frac{r+\delta+\lambda}{1+\lambda\phi_p}$$

The r.h.s, the cost of capital, increases with λ, see Lemma 1. Hence, the sign of the derivative of $\lambda(s,a,0)$ with respect to a or s is equal to the sign of the derivative of $\frac{\partial p_p y_p}{\partial k}$ with respect to a or s. We start by obtaining an expression of the marginal revenue product of capital by use of the revenue function:

$$\frac{\partial p_p y_p}{\partial k} = \frac{\sigma_p - 1}{\sigma_p}\frac{p_p y_p}{k} = \frac{\sigma_p - 1}{\sigma_p} B_p s^{\frac{\sigma_p-1}{\sigma_p}} k^{-\frac{1}{\sigma_p}}$$

where $\frac{\partial p_p y_p}{\partial k}$ declines with $k(s,a,0)$. For net worth a it is straightforward to see that $\lambda(s,a,0)$ declines with a because $k(s,a,0)$ increases with a, see Proposition 3. For the shock s we take the derivative of the marginal revenue product of capital w.r.t. s, and asking it to be non-negative delivers:

$$\frac{\partial^2 p_p y_p}{\partial k \partial s} \propto \left[(\sigma_p - 1) - \frac{\partial k}{\partial s}\frac{s}{k}\right] \geq 0$$

where the first term reflects the positive direct effect of s on the marginal revenue product of capital for fixed capital, while the second term reflects the negative indirect effect of s on the marginal revenue product of capital through its induced increase in the choice of capital. Using $\frac{\partial p_p y_p}{\partial s} = \frac{k}{s}\frac{\partial p_p y_p}{\partial k}$, equation (B.13) shows that

$$\frac{\partial k}{\partial s}\frac{s}{k} = \phi_p \frac{\partial p_p y_p}{\partial k}\left(1 - \phi_p \frac{\partial p_p y_p}{\partial k}\right)^{-1}$$

Then, we can rewrite

$$(\sigma_p - 1) - \frac{\partial k}{\partial s}\frac{s}{k} \geq 0 \quad \Leftrightarrow \quad \phi_p \frac{\partial p_p y_p}{\partial k} \leq \frac{\sigma_p - 1}{\sigma_p} \quad \Leftrightarrow \quad k \geq \phi_p p_p y_p$$

where the last step uses the fact that $\frac{\partial p_p y_p}{\partial k} = \frac{\sigma_p - 1}{\sigma_p}\frac{p_p y_p}{k}$. Note that whenever a firm has zero net worth it will be able to self-finance capital up to the point $k = \phi_p p_p y_p$. In this case the derivative of $\lambda(s, a, 0)$ with respect to s will be zero. Whenever a firm owns $a > 0$ then capital k is going to be above $\phi_p p_p y_p$ and the derivative of $\lambda(s, a, 0)$ with respect to s will be positive.

■

Next, with Corollary 1 and Proposition 4, one can also show that $\frac{\partial^2 k(s,a,0)}{\partial a^2} < 0$ (the increase in capital due to an increase in net worth is larger for firms with less net worth) and that $\frac{\partial^2 k(s,a,0)}{\partial a \partial s} > 0$ (the increase in capital due to an increase in net worth is larger for firms with higher productivity), see Corollary 2.

Corollary 2 *The derivative of $\partial k(s, a, 0)/\partial a$ with respect to a is always negative, while the derivative of $\partial k(s, a, 0)/\partial a$ with respect to s is positive as long as $a > 0$ (and zero otherwise).*

Proof: By the chain rule we can write

$$\frac{\partial^2 k(s,a,0)}{\partial a^2} = \frac{\partial^2 k(s,a,0)}{\partial a \partial \lambda}\frac{\partial \lambda(s,a,0)}{\partial a}$$
$$\frac{\partial^2 k(s,a,0)}{\partial a \partial s} = \frac{\partial^2 k(s,a,0)}{\partial a \partial \lambda}\frac{\partial \lambda(s,a,0)}{\partial s}$$

The first derivative in the r.h.s. of these expressions is positive by Corollary 1. Hence, the sign of the derivatives $\frac{\partial^2 k(s,a,0)}{\partial a^2}$ and $\frac{\partial^2 k(s,a,0)}{\partial a \partial s}$ is the same as the sign of the derivatives $\frac{\partial \lambda(s,a,0)}{\partial a}$ and $\frac{\partial \lambda(s,a,0)}{\partial s}$ described in Proposition 4. ■

Finally, we can also characterize the derivatives of the profit function $\pi(s, a, 0)$, which are given by

$$\frac{\partial \pi(s,a,0)}{\partial a} = \left[\frac{\partial p_p y_p}{\partial k} - (r + \delta)\right]\frac{\partial k(s,a,0)}{\partial a} \tag{B.16}$$
$$\frac{\partial \pi(s,a,0)}{\partial s} = \left[\frac{\partial p_p y_p}{\partial k} - (r + \delta)\right]\frac{\partial k(s,a,0)}{\partial s} + \frac{\partial p_p y_p}{\partial s} \tag{B.17}$$

We can substitute the partial derivatives of capital w.r.t. a and s described by (B.12) and (B.13) into equations (B.16) and (B.17) respectively. Then, using the FOC in (B.11) we obtain

$$\frac{\partial \pi (s,a,0)}{\partial a} = \phi_a \lambda (s,a,0) \tag{B.18}$$

$$\frac{\partial \pi (s,a,0)}{\partial s} = (1+\phi_p \lambda (s,a,0)) \frac{\partial p_p y_p}{\partial s} \tag{B.19}$$

Profits increase with a because more net worth allows to increase capital and hence profits. Profits increase with s because two reasons. First, there is the direct increase of revenues with s for given capital.

Second, if $\phi_p > 0$ a larger s implies higher revenues and hence more capital can be borrowed. Second, the increase in revenues with s allows to increase capital, which in turn increases profits. This is proved in the next Proposition:

Proposition 5 *The derivatives of $\pi (s,a,0)$ with respect to a and s are always positive.*

Proof: The derivatives of the profit function with respect to a and s are given by (B.18) and (B.19). These derivatives are positive because $\lambda (s,a,0) > 0$ for constrained agents and $\frac{\partial p_p y_p}{\partial s} > 0$ (see the revenue function). ∎

Finally, we can also characterize the second derivatives of the profit function:

Corollary 3 *The derivative of $\partial \pi (s,a,0) / \partial a$ with respect to a is always negative, while the derivative of $\partial \pi (s,a,0) / \partial s$ with respect to s is always positive as long as $a > 0$ (and zero otherwise).*

Proof: Using equation (B.18) we can write the second derivatives as,

$$\frac{\partial^2 \pi (s,a,0)}{\partial a^2} = \phi_a \frac{\partial \lambda (s,a,0)}{\partial a}$$

$$\frac{\partial^2 \pi (s,a,0)}{\partial a \partial s} = \phi_a \frac{\partial \lambda (s,a,0)}{\partial s}$$

Then, one only needs to check the signs of the derivatives of λ in Proposition 4. ∎

B.2.3 Binding constraints

Finally, we need to characterize the set of entrepreneurs that are financially constrained. Under Assumption 1, Proposition 2 says that $k(s,0,0) < k^*(s,0,0)$, and we have shown that $\frac{\partial k(s,a,0)}{\partial a} > 0$ and that $k^*(s,a,0)$ is invariant in a. Hence, for every s there will be a unique threshold $\underline{a}(s,0)$ satisfying $k(s,a,0) = k^*(s,a,0)$ such that for every s entrepreneurs with $a \geq \underline{a}(s,0)$ are unconstrained while entrepreneurs with $a < \underline{a}(s,0)$ are constrained.

B.3 Firms with procurement

We now analyze the production problem for firms with procurement, that is, firms with $d = 1$ for the case $\sigma_p = \sigma_g = \sigma$.

B.3.1 Unconstrained firms

With $\lambda = 0$ the FOC for k and u in (12) and (11) become

$$\begin{aligned} \frac{\partial p_p y_p}{\partial u} + \frac{\partial p_g y_g}{\partial u} &= 0 \\ \frac{\partial p_p y_p}{\partial k} + \frac{\partial p_g y_g}{\partial k} &= r + \delta \end{aligned}$$

which states that unconstrained firms allocate output between the two sectors to equalize the marginal revenues and choose capital such that the marginal revenue product of capital equals the capital costs. These two equations determine the optimal capital demand $k^* (s, a, 1)$ and allocation of output in the private sector $u^* (s, a, 1)$ for entrepreneurs of type $(s, a, d = 1)$. In particular, the FOC for k can be written as $\frac{\sigma-1}{\sigma} \frac{p_p y_p + p_g y_g}{k} = r + \delta$. Substituting for the revenue functions yields the optimal demand for capital:

$$k^* (s, a, 1) = \left[\left(\frac{\sigma - 1}{\sigma} \right) \frac{1}{r + \delta} \right]^{\sigma} \left(B_p^{\sigma} + B_g^{\sigma} \right) s^{\sigma - 1} \tag{B.20}$$

Using the FOC for u one gets $\frac{p_p y_p}{ku} = \frac{p_g y_g}{k(1-u)}$ where again we can substitute the revenue functions to obtain:

$$u^* (s, a, 1) = \left(1 + \left(\frac{B_g}{B_p} \right)^{\sigma} \right)^{-1} \tag{B.21}$$

Clearly $k^* (s, a, 1)$ increases monotonically with the shock s and is invariant with the net worth a, while $u^* (s, a, 1)$ is independent from both s and a and is only determined by the relative demands B_p/B_g. Next, note that profits are given by $\pi = p_p y_p + p_g y_g - (r + \delta) k$, which given the condition for the optimal choice of capital can be written as $\pi = \frac{1}{\sigma} (p_p y_p + p_g y_g)$ or $\pi = \frac{1}{\sigma - 1} (r + \delta) k$. Substituting the optimal capital demand into the revenue function gives total revenues as $p_p y_p + p_g y_g = \left[\left(\frac{\sigma-1}{\sigma} \right) \frac{1}{r+\delta} \right]^{\sigma-1} \left(B_p^{\sigma} + B_g^{\sigma} \right) s^{\sigma-1}$, which can be substituted back into the profit function to obtain

$$\pi^* (s, a, 1) = \frac{1}{\sigma} \left[\left(\frac{\sigma - 1}{\sigma} \right) \frac{1}{r + \delta} \right]^{\sigma - 1} \left(B_p^{\sigma} + B_g^{\sigma} \right) s^{\sigma - 1} \tag{B.22}$$

The profit function increases with productivity s and is invariant with assets a.

B.3.2 Constrained firms.

For constrained firms with procurement, equations (11)-(13) jointly determine $k(s,a,1)$, $u(s,a,1)$, and $\lambda(s,a,1)$. The characerization of these functions is simple whenever $\phi_g = \phi_p$ and more involved when not. To characterize $u(s,a,1)$ let's start by noting that the FOC for u, given by equation (11), can be rewritten as in (B.7) and that after substituting prices we obtain,

$$\frac{u}{1-u} = \left(\frac{B_p}{B_g}\right)^{\sigma}\left(\frac{1+\lambda\phi_p}{1+\lambda\phi_g}\right)^{\sigma} \tag{B.23}$$

To characterize $k(s,a,1)$ we totally differentiate equation (13) with respect to a and s in turn, which gives,

$$\frac{\partial k}{\partial a} = \left[\phi_a + \left(\phi_p\frac{\partial p_p y_p}{\partial u} + \phi_g\frac{\partial p_g y_g}{\partial u}\right)\frac{du}{da}\right]\left[1-\left(\phi_p\frac{\partial p_p y_p}{\partial k} + \phi_g\frac{\partial p_g y_g}{\partial k}\right)\right]^{-1} \tag{B.24}$$

$$\frac{\partial k}{\partial s} = \left[\left(\phi_p\frac{\partial p_p y_p}{\partial s} + \phi_g\frac{\partial p_g y_g}{\partial s}\right) + \left(\phi_p\frac{\partial p_p y_p}{\partial u} + \phi_g\frac{\partial p_g y_g}{\partial u}\right)\frac{du}{ds}\right]\left[1-\left(\phi_p\frac{\partial p_p y_p}{\partial k} + \phi_g\frac{\partial p_g y_g}{\partial k}\right)\right]^{-1} \tag{B.25}$$

Finally, the derivatives of the profit function $\pi(s,a,1)$ are given by

$$\begin{aligned}\frac{\partial \pi(s,a,1)}{\partial a} &= \left[\frac{\partial p_p y_p}{\partial k} + \frac{\partial p_g y_g}{\partial k} - (r+\delta)\right]\frac{\partial k(s,a,1)}{\partial a} \\ &+ \left[\frac{\partial p_p y_p}{\partial u} + \frac{\partial p_g y_g}{\partial u}\right]\frac{\partial u(s,a,1)}{\partial a}\end{aligned} \tag{B.26}$$

$$\begin{aligned}\frac{\partial \pi(s,a,1)}{\partial s} &= \left[\frac{\partial p_p y_p}{\partial k} + \frac{\partial p_g y_g}{\partial k} - (r+\delta)\right]\frac{\partial k(s,a,1)}{\partial s} \\ &+ \left[\frac{\partial p_p y_p}{\partial k} + \frac{\partial p_g y_g}{\partial k}\right]\frac{\partial u(s,a,1)}{\partial s} + \frac{\partial p_p y_p}{\partial s}\end{aligned} \tag{B.27}$$

Now, substituting (B.1), (B.2), and (B.24) into (B.26) and using (11) we obtain

$$\frac{\partial \pi(s,a,1)}{\partial a} = \phi_a \lambda(s,a,1) \tag{B.28}$$

while substituting (B.1), (B.2), and (B.25) into (B.27) and using (11) we obtain

$$\frac{\partial \pi(s,a,1)}{\partial s} = (1+\phi_p\lambda(s,a,1))\frac{\partial p_p y_p}{\partial s} + (1+\phi_g\lambda(s,a,1))\frac{\partial p_g y_g}{\partial s} \tag{B.29}$$

Profits increase with a because more net worth allows to increase capital and hence profits. Profits increase with s because two reasons. First, there is the direct increase of revenues with s for given capital. Second, if $\phi_p > 0$ and/or $\phi_g > 0$ the increase in revenues with s allows to increase capital, which in turn increases profits.

For the case $\phi_g = \phi_p$ it can be shown that $u(s,a,1) = u^*(s,a,1)$ —as revenues from both sectors are equally pledgeable— and hence $u(s,a,1)$ is invariant in a and

s. This makes the problem analogous to the case without procurement ($d = 0$), and hence the derivatives of $k(s, a, 1)$, $\lambda(s, a, 1)$, and $\pi(s, a, 1)$ with respect to a and s are as in the $d = 0$ case. This can be seen in the next propositions.

Proposition 6 *When $\phi_g = \phi_p$, the optimal choice of $u(s, a, 1)$ is as in the unconstrained case and it is hence independent from a and s*

Proof: Equation (B.23) clearly shows that whenever $\phi_g = \phi_p$ the optimal solution for u for constrained firms is equal to the one for unconstrained firms, see equation (B.21). This means that $u(s, a, 1)$ is independent from s and a and only determined by the relative demands B_p/B_g of each sector. ■

Proposition 7 *When $\phi_g = \phi_p$, the derivative of $k(s, a, 1)$ with respect to a is positive, while the derivative of $k(s, a, 1)$ with respect to s is positive as long as $\phi_p > 0$ (and zero othwewise).*

Proof: Note that with $\phi_g = \phi_p$ the optimality condition (11) implies that $\frac{\partial p_g y_g}{\partial u} = -\frac{\partial p_p y_p}{\partial u}$ and hence we can rewrite equations (B.24) and (B.25) as follows,

$$\frac{\partial k}{\partial a} = \phi_a \left[1 - \phi_p \left(\frac{\partial p_p y_p}{\partial k} + \frac{\partial p_g y_g}{\partial k}\right)\right]^{-1} \tag{B.30}$$

$$\frac{\partial k}{\partial s} = \phi_p \left(\frac{\partial p_p y_p}{\partial s} + \frac{\partial p_g y_g}{\partial s}\right)\left[1 - \phi_p \left(\frac{\partial p_p y_p}{\partial k} + \frac{\partial p_g y_g}{\partial k}\right)\right]^{-1} \tag{B.31}$$

Given $\phi_a \geq 1$ and $\phi_p > 0$ both $\partial k/\partial a$ and $\partial k/\partial s$ are positive because of Lemma 3. If $\phi_p = 0$ then $\partial k/\partial s = 0$. ■

Corollary 4 *When $\phi_g = \phi_p$, the derivatives of $k(s, a, 1)$ with respect to a and with respect to s increase with λ*

Proof: Equation (12) can be written as,

$$\frac{\partial p_p y_p}{\partial k} + \frac{\partial p_g y_g}{\partial k} = \frac{r + \delta + \lambda}{1 + \lambda \phi_p}$$

Then, using the fact that $\frac{\partial p_p y_p}{\partial s} = \frac{k}{s}\frac{\partial p_p y_p}{\partial k}$ we can rewrite equations (B.30) and (B.31) as

$$\frac{\partial k(s, a, 1)}{\partial a} = \phi_a \left(1 - \phi_p \frac{r + \delta + \lambda}{1 + \lambda \phi_p}\right)^{-1} \tag{B.32}$$

$$\frac{\partial k(s, a, 1)}{\partial s} = \phi_p \frac{k}{s} \frac{r + \delta + \lambda}{1 + \lambda \phi_p} \left(1 - \phi_p \frac{r + \delta + \lambda}{1 + \lambda \phi_p}\right)^{-1} \tag{B.33}$$

To prove this corollary it is enough to show that the term $(r + \delta + \lambda)/(1 + \lambda \phi_p)$ in equations (B.32) and (B.33) increases with λ, which is proved in Lemma 1. ■

Proposition 8 *When $\phi_g = \phi_p$, the derivative of $\lambda(s,a,1)$ with respect to a is always negative, while the derivative of $\lambda(s,a,1)$ with respect to s is always positive as long as $a > 0$ (and zero otherwise).*

Proof: Note that the FOC for k_p is given by equation (B.1). Because u is invariant in a and s, see Proposition 6, the proof of Proposition 8 for the case $d = 0$ carries over. ■

Corollary 5 *When $\phi_g = \phi_p$, the derivative of $\partial k(s,a,1)/\partial a$ with respect to a is always negative, while the derivative of $\partial k(s,a,1)/\partial a$ with respect to s is positive as long as $a > 0$ (and zero otherwise).*

Proof: By the chain rule we can write

$$\frac{\partial^2 k(s,a,1)}{\partial a^2} = \frac{\partial^2 k(s,a,1)}{\partial a \partial \lambda}\frac{\partial \lambda(s,a,1)}{\partial a}$$
$$\frac{\partial^2 k(s,a,1)}{\partial a \partial s} = \frac{\partial^2 k(s,a,1)}{\partial a \partial \lambda}\frac{\partial \lambda(s,a,1)}{\partial s}$$

The first derivative in the r.h.s. of these expressions is positive by Corollary 4. Hence, the sign of the derivatives $\frac{\partial^2 k(s,a,1)}{\partial a^2}$ and $\frac{\partial^2 k(s,a,1)}{\partial a \partial s}$ is the same as the sign of the derivatives $\frac{\partial \lambda(s,a,1)}{\partial a}$ and $\frac{\partial \lambda(s,a,1)}{\partial s}$ described in Proposition 8. ■

Proposition 9 *When $\phi_g = \phi_p$, the derivatives of $\pi(s,a,1)$ with respect to a and s are always positive.*

Proof: The derivatives of the profit function with respect to a and s are given by (B.28) and (B.29). These derivatives are positive because $\lambda(s,a,1) > 0$ for constrained agents and $\frac{\partial p_p y_p}{\partial s} > 0$ and $\frac{\partial p_g y_g}{\partial s} > 0$ (see the revenue functions). ■

Corollary 6 *When $\phi_g = \phi_p$, the derivative of $\partial \pi(s,a,1)/\partial a$ with respect to a is always negative, while the derivative of $\partial \pi(s,a,1)/\partial s$ with respect to s is always positive as long as $a > 0$ (and zero otherwise).*

Proof: Using equation (B.28) we can write the second derivatives as,

$$\frac{\partial^2 \pi(s,a,1)}{\partial a^2} = \phi_a \frac{\partial \lambda(s,a,1)}{\partial a}$$
$$\frac{\partial^2 \pi(s,a,1)}{\partial a \partial s} = \phi_a \frac{\partial \lambda(s,a,1)}{\partial s}$$

Then, one only needs to check the signs of the derivatives of λ in Proposition 8. ■

The case $\phi_g > \phi_p$ is more involved because $u(s,a,1)$ changes with a and s. It can be shown that firms with more net worth are less constrained and hence run larger firms and sell a higher fraction of output to the private sector, which offers lower collateral value. More productive firms are able to run larger firms thanks to the earnings-based constraints but are more constrained —because their optimal capital is even larger— and hence sell a lower fraction of output to the private sector. This which means that firms with larger s sell a larger quantity to the pubic sector but they may either sell a larger or smaller quantity to the private sector. This is proved in the following propositions.

Lemma 5 *The sign of the derivative of u with respect to λ is the same as the sign of $(\phi_p - \phi_g)$, that is, more constrained firms shift their output relatively towards the sector whose revenues provide better colateral.*

Proof: Simply note that equation (B.23) implies that $du/d\lambda < 0$ when $\phi_g > \phi_p$ and the opposite when $\phi_g < \phi_p$. ■

Proposition 10 *When $\phi_g > \phi_p$, the derivatives of $u(s, a, 1)$, $k(s, a, 1)$, and $\lambda(s, a, 1)$ with respect to a are positive, positive, and negative respectively,*

Proof: First note that, following Lemma 5, $du/d\lambda < 0$ when $\phi_g > \phi_p$ and that Proposition 1 says that $dk/d\lambda < 0$. That is, more constrained entrepreneurs tilt production towards the sector with higher collateral value and run smaller firms. Next, using the FOC (B.3) and (B.4), the demand equations (4), and the production function we can write,

$$k_p = \left(\frac{\sigma_p - 1}{\sigma_p} B_p \frac{1 + \lambda\phi_p}{r + \delta + \lambda}\right)^{\sigma_p} s^{\sigma_p - 1} \quad \text{and} \quad k_g = \left(\frac{\sigma_p - 1}{\sigma_p} B_g \frac{1 + \lambda\phi_g}{r + \delta + \lambda}\right)^{\sigma_p} s^{\sigma_p - 1}$$

Adding them up, and using the chain rule, let us express $\frac{\partial k}{\partial a}$

$$\frac{\partial k}{\partial a} = \frac{\partial k}{\partial \lambda}\frac{\partial \lambda}{\partial a}$$

Also, using equation (B.23) and the chain rule we can write

$$\frac{\partial u}{\partial a} = \frac{\partial u}{\partial \lambda}\frac{\partial \lambda}{\partial a}$$

These two expressions state that $\frac{\partial k}{\partial a}$ and $\frac{\partial u}{\partial a}$ should have the same sign because both k and u fall with λ. Given this, equation (B.24) implies that $\frac{\partial k}{\partial a} > 0$ and $\frac{\partial u}{\partial a} > 0$. To see why, recall that by Lemma 3 the denominator is positive. In addition, the term $\phi_p \frac{\partial p_p y_p}{\partial u} + \phi_g \frac{\partial p_g y_g}{\partial u}$ is negative whenever $\phi_g > \phi_p$ see Lemma 4. Hence, for $\frac{\partial k}{\partial a} < 0$ we would need $\frac{\partial k}{\partial u} > 0$. That is, given that higher a allows to increase capital through ϕ_a, for higher a to lead to lower capital it must be that entrepreneurs with higher a tilt production towards the sector with lower collateral value. But this would require the signs of $\frac{\partial k}{\partial a}$ and $\frac{\partial u}{\partial a}$ to be different. Instead, $\frac{\partial k}{\partial a} > 0$ can be obtained with $\frac{\partial u}{\partial a} > 0$. It follows that, because $\frac{\partial k}{\partial \lambda} < 0$ and $\frac{\partial k}{\partial a} > 0$, it must be the case that $\frac{\partial \lambda}{\partial a} < 0$. ■

Proposition 11 *When $\phi_g > \phi_p$, the derivatives of $u(s,a,1)$, $k(s,a,1)$, and $\lambda(s,a,1)$ with respect to s are negative, positive, and positive respectively,*

Proof: First note that, following Lemma 5, $du/d\lambda < 0$ when $\phi_g > \phi_p$ and that Proposition 1 says that $dk/d\lambda < 0$. That is, more constrained entrepreneurs tilt production towards the sector with higher collateral value and run smaller firms. Next, by the chain rule (see proof of Proposition 10) we can write

$$\frac{dk}{ds} = \frac{\partial k}{\partial \lambda}\frac{\partial \lambda}{\partial s} + \frac{\partial k}{\partial s} \quad \text{and} \quad \frac{du}{ds} = \frac{\partial u}{\partial \lambda}\frac{\partial \lambda}{\partial s}$$

We learn two things from here. First, $\frac{dk}{ds} \le 0$ requires $\frac{\partial \lambda}{\partial s} > 0$ (because $\frac{\partial k}{\partial s} > 0$ and $\frac{\partial k}{\partial \lambda} < 0$). Second, $\frac{\partial \lambda}{\partial s} > 0$ requires $\frac{du}{ds} < 0$ (because $du/d\lambda < 0$). But equation (B.25) shows that if $\frac{du}{ds} < 0$ then it must be $\frac{dk}{ds} > 0$ so this enters a contradiction. Therefore, $\frac{dk}{ds} > 0$. Note that from equation (B.25) $\frac{dk}{ds} > 0$ can be achieved with any sign of $\frac{du}{ds}$. Now, regarding the derivatives of $u(s,a,1)$ and $\lambda(s,a,1)$ with respect to s, two different things can happen. If $\frac{\partial \lambda}{\partial s} \ge 0$ then $\frac{du}{ds} \le 0$ (this is an if and only if statement), and then $\frac{dk}{ds} > 0$ according to equation (B.25). Instead, if $\frac{\partial \lambda}{\partial s} < 0$ then $\frac{du}{ds} > 0$ (again an if and only if statement) and we can have both $\frac{dk}{ds} > 0$ or $\frac{dk}{ds} < 0$ according to equation (B.25). ■

Proposition 12 *When $\phi_g > \phi_p$, the derivatives of $\pi(s,a,1)$ with respect to a and s are always positive.*

Proof: The derivatives of the profit function with respect to a and s are given by (B.28) and (B.29). These derivatives are positive because $\lambda(s,a,1) > 0$ for constrained agents and $\frac{\partial p_p y_p}{\partial s} > 0$ and $\frac{\partial p_g y_g}{\partial s} > 0$ (see the revenue functions). ■

Corollary 7 *When $\phi_g > \phi_p$, the derivative of $\partial\pi(s,a,1)/\partial a$ with respect to a is always negative, while the derivative of $\partial\pi(s,a,1)/\partial s$ with respect to s is always positive as long as $a > 0$ (and zero otherwise).*

Proof: Using equation (B.28) we can write the second derivatives as,

$$\begin{aligned}\frac{\partial^2\pi(s,a,1)}{\partial a^2} &= \phi_a \frac{\partial \lambda(s,a,1)}{\partial a}\\ \frac{\partial^2\pi(s,a,1)}{\partial a \partial s} &= \phi_a \frac{\partial \lambda(s,a,1)}{\partial s}\end{aligned}$$

Then, one only needs to check the signs of the derivatives of λ in Proposition 10 and 11. ■

B.4 A procurement shock

Finally, in this Section we analyze how firm choices change upon arrival of a procurement project for the case $\sigma_p = \sigma_g = \sigma$. To do so, we compare the choices of firms in the $(s,a,1)$ state with firms in the $(s,a,0)$ state.

B.4.1 Unconstrained firms

For unconstrained firms, the increase in total capital is given by,

$$\frac{k^*(s,a,1)}{k^*(s,a,0)} = 1 + \left(\frac{B_g}{B_p}\right)^\sigma = \frac{1}{u^*(s,a,1)}$$

which implies that $u^*(s,a,1)\,k^*(s,a,1) = k^*(s,a,0)$. Hence, the amount of capital used in the private sector for the unconstrained firm with a procurement project equals the capital stock it was using without procurement. This means that unconstrained firms do not change their private sector operations and increase their capital stock to meet the extra demand. The increase in capital $k^*(s,a,1) - k^*(s,a,0)$ is given by $\left(\frac{B_g}{B_p}\right)^\sigma k^*(s,a,0)$. Because $k^*(s,a,0)$ increases with s and is independent from a, so does the capital increase with procurement.

We can also see that the value of a procurement contract increases with firm productivity s and is independent from firm net worth a. This can be seen by use of the expression $\pi = \frac{1}{\sigma-1}(r+\delta)\,k$, which implies that $\pi^*(s,a,1) - \pi^*(s,a,0)$ is proportional to the capital increase $k^*(s,a,1) - k^*(s,a,0)$. This could have also be seen by combining equations (B.9) and (B.22), which allows to express

$$\pi^*(s,a,1) - \pi^*(s,a,0) = \frac{1}{\sigma}\left[\left(\frac{\sigma-1}{\sigma}\right)\frac{1}{r+\delta}\right]^{\sigma-1} B_g^\sigma s^{\sigma-1}$$

B.4.2 Constrained firms

The first thing to note is that a procurement shock worsens the financial situation of firms when $\phi_g \le \phi_p$. With $\phi_g = \phi_p$ this is because the firm with $d = 1$ has two demands to serve, they are equally pledgeable, and has the same net worth a to finance capital in the two different markets. As a result the firm scales down the operations in the private sector to free up colateral for the production in the public sector, which generates a negative within-firm private sector spillover of the procurement contract, that is, $k_p(s,a,1) \equiv u(s,a,1)\,k(s,a,1) < k(s,a,0)$. When $\phi_g < \phi_p$ the situation is aggravated because the public sector demand can be self-financed to a lesser extent. When $\phi_g > \phi_p$ it could happen otherwise: the public sector demand can be self-financed to a larger extent, which means that for firms with small net worth it could happen that they are less constrained and use the extra financing capacity coming from the public sector to scale up operations in the private sector. This is stated in Proposition 13 below, but we first look at two preliminary results in Lemma 6 and 7.

Lemma 6 *A procurement shock generates a private sector negative spillover if and only if the procurement shock makes the firm more constrained, that is,* $k_p(s,a,1) < k(s,a,0) \Leftrightarrow \lambda(s,a,1) > \lambda(s,a,0)$

Proof: The FOC for the optimal choice of k_p for a firm with $d = 1$ is given by equation (B.1), where recall $\frac{\partial p_p y_p}{\partial k}\frac{1}{u} = \frac{\partial p_p y_p}{\partial k_p}$. The FOC for the optimal choice of k for a firm with $d = 0$ is given by the same equation (B.1) when $u = 1$. The right hand side of equation (B.1) increase with λ (see Lemma 1), so more constrained firms have higher marginal product of capital and a lower level of capital in the private sector. Hence, $k_p(s,a,1) < k(s,a,0) \Leftrightarrow \lambda(s,a,1) > \lambda(s,a,0)$ ■

Lemma 7 *A procurement shock generates a private sector negative spillover for constrained firms if and only if the chosen production for the public sector cannot be self-financed, that is, if and only if* $\phi_g p_g(s,a,1) y_g(s,a,1) < k_g(s,a,1)$

Proof: The demand for capital of constrained firms, with or without procurement, is given by equation (9), which allows to write,

$$\begin{aligned} k_p(s,a,0) - \phi_p p_p(s,a,0) y_p(s,a,0) &= \phi_a a \\ k_p(s,a,1) - \phi_p p_p(s,a,1) y_p(s,a,1) &= \phi_a a - [k_g(s,a,1) - \phi_g p_g(s,a,1) y_g(s,a,1)] \end{aligned}$$

Importantly, the left hand side of these equations increases with k_p. To see how, note that the derivative of the left hand side w.r.t. k_p is equal to $1 - \phi_p \frac{\partial p_p y_p}{\partial k_p} = 1 - \phi_p \frac{r+\delta+\lambda}{1+\lambda\phi_p}$ according to equation (B.1). Now, $\phi_p \frac{r+\delta+\lambda}{1+\lambda\phi_p} < 1$ according to Lemma 1, so the derivative is positive. Hence, if $k_p(s,a,1) < k_p(s,a,0)$ then $[k_p(s,a,1) - \phi_p p_p(s,a,1) y_p(s,a,1)] < [k_p(s,a,0) - \phi_p p_p(s,a,0) y_p(s,a,0)]$ which requires $\phi_g p_g(s,a,1) y_g(s,a,1) < k_g(s,a,1)$.
■

Proposition 13 *When* $\phi_g \leq \phi_p$, *a procurement shock for constrained firms generates a private sector negative spillover, that is,* $k_p(s,a,1) < k(s,a,0)$, *makes the firm more constrained, that is,* $\lambda(s,a,1) > \lambda(s,a,0)$, *and production in the government sector cannot be self-financed, that is,* $\phi_g p_g(s,a,1) y_g(s,a,1) < k_g(s,a,1)$. *When* $\phi_g > \phi_p$ *the same will happen, with the exception of firms with very small net worth for which the opposite will happen.*

Proof: To prove the first part, let's rewrite the borrowing constraint in (9) for $d = 0$ firms as

$$1 = \phi_a \frac{a}{k(s,a,0)} + \phi_p \frac{p_p(s,a,0) y_p(s,a,0)}{k(s,a,0)} \tag{B.34}$$

and for $d = 1$ firms as

$$\begin{aligned} 1 &= \phi_a \frac{a}{k_p(s,a,1) + k_g(s,a,1)} + \phi_p \frac{p_p(s,a,1) y_p(s,a,1)}{k_p(s,a,1)} \\ &+ (1 - u(s,a,1)) \left[\phi_g \frac{p_g(s,a,1) y_g(s,a,1)}{k_g(s,a,1)} - \phi_p \frac{p_p(s,a,1) y_p(s,a,1)}{k_p(s,a,1)} \right] \end{aligned} \tag{B.35}$$

If $\phi_g = \phi_p$, firms with $d = 1$ equalize the average product in the public and private sectors, see equations (B.3) and (B.4), so that the third term in equation (B.35) disappears. In this case, if $k_g(s,a,1) = 0$ then equations (B.34) and (B.35) are identical and $k_p(s,a,1) = k(s,a,0)$. However, because the marginal revenue product in the public sector goes to infinity when $k_g(s,a,1) = 0$, it must be that $k_g(s,a,1) > 0$ and hence comparison of equations (B.34) and (B.35) requires $k_p(s,a,1) < k(s,a,0)$. If $\phi_g < \phi_p$, then the third term in equation (B.35) is negative. This can be easily seen by multiplying both sides of equation (B.3) by ϕ_p and both sides of equation (B.4) by ϕ_g. Then whenever $k_g > 0$ and hence $(1-u) > 0$, equation (B.35) requires $k_p(s,a,1) < k(s,a,0)$ to hold. The second and third parts of the Proposition come from Lemma 6 and Lemma 7 respectively. Finally, for the case $\phi_g > \phi_p$ the third term in equation (B.35) is positive. If $a = 0$ this requires $k_p(s,a,1) > k(s,a,0)$ for equation (B.35) to hold as the first term in the right hand side of equation (B.35) disappears. For $a > 0$, the first term in the right hand side of equation (B.35) reappears and offsets this force. More specifically, as a increases, λ falls by Proposition 10, and thus $\frac{p_g(s,a,1)y_g(s,a,1)}{k_g(s,a,1)}$ decreases. In the limit, if a becomes sufficiently large and exceeds the net worth level $a_g^*(s)$ above which the procurement firm is unconstrained, $\frac{p_g(s,a,1)y_g(s,a,1)}{k_g(s,a,1)}$ falls to the unconstrained level of $\frac{\sigma}{\sigma-1}(r+\delta)$, which is strictly smaller than $\frac{1}{\phi_g}$ by Assumption 1. This means that there exists a cutoff level $\bar{a}_g(s)$ such that if $a \in (\bar{a}_g(s), a_g^*(s))$, then $\phi_g p_g(s,a,1)y_g(s,a,1) < k_g(s,a,1)$. And by Lemma 7, this means that the spillover is negative for a in this interval. ■

Proposition 14 *Whenever $\phi_g \geq \phi_p > 0$ having access to procurement always generates an increase in firm size, that is, $k(s,a,1) > k(s,a,0)$ $\forall a, s$. Whenever $\phi_g < \phi_p$ the opposite may happen. In the particular case that $\phi_g = \phi_p = 0$ a procurement shock does not change the size of the firm.*

Proof: We prove the $\phi_g \geq \phi_p > 0$ case by contradiction by showing that if $k(s,a,1) \leq k(s,a,0)$, then the borrowing constraint for the firm with $d = 1$ would not bind, which could not be optimal; so it must be that $k(s,a,1) > k(s,a,0)$. To se why, we start with the case $k(s,a,1) = k(s,a,0)$. In this situation, the firm with $d = 1$ optimally chooses $u(s,a,1) < 1$ because the marginal revenue product of revenues in the public sector tend to infinity as u tends to 1. This generates more revenues and because $\phi_g \geq \phi_p > 0$, Lemma 4 guarantees that this also generates more (unused) borrowing capacity, so it cannot be optimal. If $k(s,a,1) < k(s,a,0)$ and $u(s,a,1) = 1$ this again generates slack in the borrowing constraint because of Lemma 3, and cannot be optimal. But lowering u generates the same or further slack when $\phi_g \geq \phi_p > 0$, see Lemma 4. So $k(s,a,1) < k(s,a,0)$ cannot be optimal either. Note that the argument by contradiction requires that $\phi_g \geq \phi_p > 0$ such that when the firm with $d = 1$ substitutes private revenues with public revenues the borrowing capacity increases. When $\phi_g < \phi_p$, instead, the contrary happens because selling to the government limits the borrowing capacity of the firm, and the proof does not hold. For example, it can be shown that with $0 = \phi_g < \phi_p$ we will have $k(s,a,1) < k(s,a,0)$. Using the financial constraint, the difference in the capital that can be financed with $d = 1$ and $d = 0$

when $\phi_g = 0$ is given by,

$$k(s,a,1) - k(s,a,0) = \phi_p \left[p_p(s,a,1)\, y_p(s,a,1) - p_p(s,a,0)\, y_p(s,a,0) \right]$$

Proposition 13 says that there is a negative private sector spillover, that is $p_p(s,a,1)\, y_p(s,a,1) < p_p(s,a,0)\, y_p(s,a,0)$, whenever $\phi_g < \phi_p$, so we will have $k(s,a,1) < k(s,a,0)$. Finally, note that with $\phi_g = \phi_p = 0$, $k(s,a,1) = k(s,a,0)$ as capital for constrained firms is determined only by a. ■

Proposition 15 *Having access to procurement always generates extra profits, that is, $\pi(s,a,1) > \pi(s,a,0)$ $\forall s, a$. Whenever $\phi_g \leq \phi_p$, the value of procurement is increasing in net worth; whenever $\phi_g > \phi_p$, the value of procurement is generally increasing in net worth except for firms with very low net worth when the opposite will happen. The value of procurement is increasing in firm productivity whenever $\phi_g \geq \phi_p$.*

Proof: The first part is trivial. A firm with $d = 1$ has profits equal to

$$\pi(s,a,1) = p_p(s,a,1)\, y_p(s,a,1) + p_g(s,a,1)\, y_g(s,a,1) - (r+\delta)\, k(s,a,1)$$

and can always replicate the profits of a firm with $d = 0$ by choosing $u(s,a,1) = 1$. Because of our functional form assumptions, the marginal revenue product of capital in the public sector, $\partial p_g y_g / \partial k_g$, tends to infinity whenever $u(s,a,1) = 1$, so it means that it is optimal for any firm with $d = 1$ to choose $u(s,a,1) < 1$ and increase profits compared to the case $u(s,a,1) = 1$ and therefore compared to the case of no procurement. For the second part we want to show that $\frac{[\partial \pi(s,a,1) - \partial \pi(s,a,0)]}{\partial a} > 0$. Equations (B.18) and (B.28) imply

$$\frac{\partial \left[\pi(s,a,1) - \pi(s,a,0) \right]}{\partial a} = \phi_a \Big[\lambda(s,a,1) - \lambda(s,a,0) \Big] > 0$$

and the sign of $\lambda(s,a,1) - \lambda(s,a,0)$ is given by Proposition 13. Finally, for the third part we want to show that $\frac{[\partial \pi(s,a,1) - \partial \pi(s,a,0)]}{\partial s} > 0$ whenever $\phi_g \geq \phi_p$. Equations (B.19) and (B.29) imply

$$\frac{[\partial \pi(s,a,1) - \partial \pi(s,a,0)]}{\partial s} = (1 + \phi_p \lambda(s,a,1)) \frac{\partial p_p y_p}{\partial s} + (1 + \phi_g \lambda(s,a,1)) \frac{\partial p_g y_g}{\partial s} - (1 + \phi_p \lambda(s,a,0)) \frac{\partial p_p}{\partial s}$$

Note that $\frac{\partial p_p y_p}{\partial s} = \frac{k_p}{s} \frac{\partial p_p y_p}{\partial k_p} = \frac{k_p}{s} \frac{r+\delta+\lambda}{1+\lambda \phi_p}$ and an analogous expression holds for the public good. Substituting these expressions in the above equation gives

$$\frac{[\partial \pi(s,a,1) - \partial \pi(s,a,0)]}{\partial s} = \frac{r + \delta + \lambda(s,a,1)}{s} \left[k_p(s,a,1) + k_g(s,a,1) \right] - \frac{r + \delta + \lambda(s,a,0)}{s} k_p(s,a,0)$$

With $\phi_g \geq \phi_p$, Proposition 14 states that $k_p(s,a,1) + k_g(s,a,1) > k_p(s,a,0)$. Therefore, whenever $\lambda(s,a,1) > \lambda(s,a,0)$ we can guarantee that $\frac{[\partial \pi(s,a,1) - \partial \pi(s,a,0)]}{\partial s} > 0$. According to Proposition 13 this will generally happen, except for very low a when

$\lambda(s,a,1) < \lambda(s,a,0)$. However, in this case we can still show the statement to be true by showing that $\frac{r+\delta+\lambda(s,a,1)}{s}k_p(s,a,1) > \frac{r+\delta+\lambda(s,a,0)}{s}k_p(s,a,0)$. To show this, we take the FOC for k_p in equation (B.1) to obtain an expression for λ as:

$$\lambda = \frac{\frac{\partial p_p y_p}{\partial k_p} - (r+\delta)}{1-\phi_p \frac{\partial p_p y_p}{\partial k_p}}$$

Then adding $(r+\delta)$ in both sides, rearranging, and multiplying by k_p in both sides we obtain

$$(r+\delta+\lambda)k_p = [1-\phi_p(r+\delta)]\frac{\frac{\partial p_p y_p}{\partial k_p}}{1-\phi_p \frac{\partial p_p y_p}{\partial k_p}}k_p$$

Using our functional form for the revenue function, we can rewrite the last terms as:

$$\frac{\frac{\partial p_p y_p}{\partial k_p}}{1-\phi_p \frac{\partial p_p y_p}{\partial k_p}}k_p = \frac{\sigma-1}{\sigma}\frac{B_p s^{\frac{\sigma-1}{\sigma}} k_p^{\frac{\sigma-1}{\sigma}}}{1-\phi_p \frac{\sigma-1}{\sigma} B_p s^{\frac{\sigma-1}{\sigma}} k_p^{\frac{-1}{\sigma}}}$$

Taking the derivative of this object w.r.t. k_p, we have:

$$\begin{aligned}
\frac{\partial}{\partial k_p}\left(\frac{\frac{\partial p_p y_p}{\partial k_p}}{1-\phi_p \frac{\partial p_p y_p}{\partial k_p}}k_p\right) &\propto \frac{\sigma-1}{\sigma}B_p s^{\frac{\sigma-1}{\sigma}} k_p^{\frac{-1}{\sigma}}\left[1-\phi_p\frac{\sigma-1}{\sigma}B_p s^{\frac{\sigma-1}{\sigma}} k_p^{\frac{-1}{\sigma}}\right] - B_p s^{\frac{\sigma-1}{\sigma}} k_p^{\frac{\sigma-1}{\sigma}}\frac{\sigma-1}{\sigma}\frac{1}{\sigma}\phi_p B_p s^{\frac{\sigma-1}{\sigma}} k_p^{\frac{-1}{\sigma}-1} \\
&= \frac{\sigma-1}{\sigma}B_p s^{\frac{\sigma-1}{\sigma}} k_p^{\frac{-1}{\sigma}}\left\{\left[1-\phi_p\frac{\sigma-1}{\sigma}B_p s^{\frac{\sigma-1}{\sigma}} k_p^{\frac{-1}{\sigma}}\right] - \phi_p\frac{1}{\sigma}B_p s^{\frac{\sigma-1}{\sigma}} k_p^{\frac{-1}{\sigma}}\right\} \\
&= \frac{\sigma-1}{\sigma}B_p s^{\frac{\sigma-1}{\sigma}} k_p^{\frac{-1}{\sigma}}\left\{1-\phi_p B_p s^{\frac{\sigma-1}{\sigma}} k_p^{\frac{-1}{\sigma}}\right\} \\
&= \frac{\sigma-1}{\sigma}B_p s^{\frac{\sigma-1}{\sigma}} k_p^{\frac{-1}{\sigma}}\left\{1-\phi_p\frac{p_p y_p}{k_p}\right\} > 0
\end{aligned}$$

where the last inequality follows from the fact that by Lemma 3, $\phi_p \frac{p_p y_p}{k_p} < 1$ for constrained firms. This establishes that for constrained firms, the term $(r+\delta+\lambda)k_p$ must be higher whenever k_p is higher. Therefore, if $\lambda(s,a,1) < \lambda(s,a,0)$, the k_p FOC, implies that $k_p(s,a,1) > k_p(s,a,0)$, which in turns implies $[r+\delta+\lambda(s,a,1)]k_p(s,a,1) > [r+\delta+\lambda(s,a,0)]k_p(s,a,0)$. And trivially, this implies $\frac{r+\delta+\lambda(s,a,1)}{s}[k_p(s,a,1)+k_g(s,a,1)] - \frac{r+\delta+\lambda(s,a,0)}{s}k_p(s,a,0)$, proving the statement whenever $\phi_g \geq \phi_p$ and $\lambda(s,a,1) < \lambda(s,a,0)$.
■

C Details on some aggregates

C.1 Equilibrium definition

Let $\mathbf{X} \equiv S \times A \times \{0,1\}$ be the state space of the household problem, $\mathbf{X}_1 \equiv S \times A \times \{1\}$ the subset of the state space for firms with a procurement project, $\mathcal{X}$ a σ-algebra generated by $\mathbf{X}$, and Γ a probability measure over $\mathcal{X}$. Then, given government policy parameters Y_g and m_g and a distribution of entrants Γ_0, the steady state equilibrium requires:

a) Entrepreneurs solve their optimization problem

b) The probability measure Γ is stationary

c) The market for the private good clears:

$$\int_{\mathbf{X}} p_p(s,a,d)\, u(s,a,d)\, y(s,a,d)\, d\Gamma = Y_p = \int_{\mathbf{X}} \left[b(s,a,d) + c(s,a,d) + \delta k(s,a,d)\right] d\Gamma$$

d) The market for the public good clears:

$$\int_{\mathbf{X}_1} p_g(s,a,1)\left[1 - u(s,a,1)\right] y(s,a,1)\, d\Gamma = P_g Y_g$$

e) The probability of obtaining procurement projects is consistent with the measure of goods bought by the public sector,

$$\int_{\mathbf{X}} Pr\left(d' = 1 \mid b(s,a,d)\right) d\Gamma = \int_{\mathbf{X}_1} d\Gamma = m_g$$

f) The budget constraint of the government holds

$$P_g Y_g = rD + \tau \int_{\mathbf{X}} \pi(s,a,d)\, d\Gamma + (1-\theta)\left[\int_{\mathbf{X}} a'(s,a,d) d\Gamma - \int_{\mathbf{X}} a d\Gamma_0\right]$$

g) By Walras law, the credit market clears.

$$D = \int_{\mathbf{X}} \left[k(s,a,d) - a(s,a,d)\right] d\Gamma$$

C.2 Sectorial and aggregate TFP

The TFP for the private and public sectors are given by,

$$\text{TFP}_p \equiv \frac{Y_p}{K_p} = \left[\int_{[0,1]} \left(s_i \frac{\overline{\text{MRPK}}_p}{\text{MRPK}_{ip}}\right)^{\sigma_p - 1} di\right]^{\frac{1}{\sigma_p - 1}}, \quad \text{TFP}_g \equiv \frac{Y_g}{K_g} = \left[\int_{I_g} \frac{1}{m_g}\left(s_i \frac{\overline{\text{MRPK}}_g}{\text{MRPK}_{ig}}\right)^{\sigma_g - 1} di\right]^{\frac{1}{\sigma_g - 1}} \tag{C.1}$$

where

$$\frac{1}{\overline{\text{MRPK}}_p} \equiv \int_{[0,1]} \frac{p_{ip} y_{ip}}{P_p Y_p} \frac{1}{\text{MRPK}_{ip}} di, \quad \frac{1}{\overline{\text{MRPK}}_g} \equiv \int_{I_g} \frac{1}{m_g} \frac{p_{ig} y_{ig}}{P_g Y_g} \frac{1}{\text{MRPK}_{ig}} di \tag{C.2}$$

Then aggregate TFP = $(Y_p + P_g Y_g)/(K_p + K_g)$ in units of the private sector good is given by the weighted average

$$\text{TFP} = \text{TFP}_p \frac{K_p}{K_p + K_g} + P_g \text{TFP}_g \frac{K_g}{K_p + K_g} \tag{C.3}$$

Finally, absent financial frictions there would be no heterogeneity in $\overline{\text{MRPK}}_p$ and $\overline{\text{MRPK}}_g$ and optimal TFP in the private and public sectors (conditional on selection) would be,

$$\text{TFP}_p^* = \left[\int_{[0,1]} s_i^{\sigma_p - 1} di\right]^{\frac{1}{\sigma_p - 1}} \quad \text{and} \quad \text{TFP}_g^* = \left[\int_{I_g} \frac{1}{m_g} s_i^{\sigma_g - 1} di\right]^{\frac{1}{\sigma_g - 1}} \tag{C.4}$$

C.3 Relative price of public sector good

Using the definitions of P_g and P_p in equations (5), the relative price can be written as,

$$\frac{P_g}{P_p} = \frac{\left[\int_{I_g} \frac{1}{m_g} p_{ig}^{1-\sigma_g} di\right]^{\frac{1}{1-\sigma_g}}}{\left[\int_{[0,1]} \quad p_{ip}^{1-\sigma_p} di\right]^{\frac{1}{1-\sigma_p}}} = \frac{\left[\int_{I_g} \frac{1}{m_g} \left(\frac{1}{s_i} \text{MRPK}_{ig}\right)^{1-\sigma_g} di\right]^{\frac{1}{1-\sigma_g}}}{\left[\int_{[0,1]} \quad \left(\frac{1}{s_i} \text{MRPK}_{ip}\right)^{1-\sigma_p} di\right]^{\frac{1}{1-\sigma_p}}}$$

where the last equality follows from using the definition of MRPK_{ip} and the production function as follows,

$$\text{MRPK}_{ip} \equiv \frac{\partial p_{ip} y_{ip}}{\partial k_{ip}} = \frac{\sigma_p - 1}{\sigma_p} \frac{p_{ip} y_{ip}}{k_{ip}} = \frac{\sigma_p - 1}{\sigma_p} p_{ip} s_i \quad \Rightarrow \quad p_{ip} = \frac{\sigma_p}{\sigma_p - 1} \frac{1}{s_i} \text{MRPK}_{ip}$$

and the same applies for MRPK_{ig}. Next multiplying and dividing by $\overline{\text{MRPK}}_g$ in the numerator and by $\overline{\text{MRPK}}_p$ in the denominator we obtain,

$$\frac{P_g}{P_p} = \frac{\overline{\text{MRPK}}_g}{\overline{\text{MRPK}}_p} \frac{\left[\int_{I_g} \frac{1}{m_g} \left(\frac{1}{s_i} \frac{\overline{\text{MRPK}}_g}{\text{MRPK}_{ig}}\right)^{1-\sigma_g} di\right]^{\frac{1}{\sigma_g - 1}}}{\left[\int_{[0,1]} \quad \left(\frac{1}{s_i} \frac{\overline{\text{MRPK}}_p}{\text{MRPK}_{ip}}\right)^{\sigma_p - 1} di\right]^{\frac{1}{1-\sigma_p}}} = \frac{\overline{\text{MRPK}}_p}{\overline{\text{MRPK}}_g} \frac{\text{TFP}_p}{\text{TFP}_g}$$

C.4 Relative sectoral TFP

Given the definition of TFP_p in equation (C.1), we can write

$$\begin{aligned} \text{TFP}_p &= \left[m_g \int_{I_g} \frac{1}{m_g} \left(s_i \frac{\overline{\text{MRPK}}_p}{\text{MRPK}_{ip}}\right)^{\sigma_p - 1} di + (1 - m_g) \int_{I_g^c} \frac{1}{1 - m_g} \left(s_i \frac{\overline{\text{MRPK}}_p}{\text{MRPK}_{ip}}\right)^{\sigma_p - 1} di\right]^{\frac{1}{\sigma_p - 1}} \\ &= \left[m_g \text{TFP}_{p,I_g}^{\sigma_p - 1} + (1 - m_g) \text{TFP}_{p,I_g^c}^{\sigma_p - 1}\right]^{\frac{1}{\sigma_p - 1}} \end{aligned}$$

where we have defined TFP_{p,I_g} and TFP_{p,I_g^c} as the average TFP in the private sector within the set of procurement (I_g) and non-procurement (I_g^c) firms respectively. Then, dividing by TFP_g in both sides we get the expression for $\text{TFP}_p/\text{TFP}_g$:

$$\frac{\text{TFP}_p}{\text{TFP}_g} = \left[m_g \left(\frac{\text{TFP}_{p,I_g}}{\text{TFP}_g} \right)^{\sigma_p - 1} + (1 - m_g) \left(\frac{\text{TFP}_{p,I_g^c}}{\text{TFP}_g} \right)^{\sigma_p - 1} \right]^{\frac{1}{\sigma_p - 1}} \tag{C.5}$$

The first term in equation (C.5) reflects the within-firm misallocation. With $\sigma_g = \sigma_p$ this term would be equal to 1 if $\phi_g = \phi_p$ or if there were no financial frictions ($\lambda_i = 0 \; \forall i$). Instead, if $\phi_g > \phi_p$ firms switch their output relatively towards the public sector and the dispersion of MRPK_{ig} declines, which makes $\text{TFP}_{p,I_g}/\text{TFP}_g$ fall. The second term in equation (C.5) reflects both between-firm misallocation and selection into procurement. If firms with higher s self-select into procurement, then $\text{TFP}_{p,I_g^c}/\text{TFP}_g$ declines. If there is more dispersion in MRPK_{ip} between non-procurement firms than in MRPK_{ig} between procurement firms, then $\text{TFP}_{p,I_g^c}/\text{TFP}_g$ is lower. In short, absent financial frictions the only reason for $\text{TFP}_p/\text{TFP}_g \neq 1$ would be the selection of firms into procurement. In the first best (no financial frictions and the government selects the firms with highest s) we would have $\text{TFP}_p/\text{TFP}_g < 1$.

C.5 Relative sectoral $\overline{\text{MRPK}}$

Given the definition of $\overline{\text{MRPK}}_p$ in equation (C.2), we can write

$$\begin{aligned} \overline{\text{MRPK}}_p &= \left[\frac{R_{p,I_g}}{P_p Y_p} \int_{I_g} \frac{p_{ip} y_{ip}}{R_{p,I_g}} \text{MRPK}_{ip}^{-1} di + \frac{R_{p,I_g^c}}{P_p Y_p} \int_{I_g^c} \frac{p_{ip} y_{ip}}{R_{p,I_g^c}} \text{MRPK}_{ip}^{-1} di \right]^{-1} \\ &= \left[\frac{R_{p,I_g}}{P_p Y_p} \overline{\text{MRPK}}_{p,I_g}^{-1} + \frac{R_{p,I_g^c}}{P_p Y_p} \overline{\text{MRPK}}_{p,I_g^c}^{-1} \right]^{-1} \end{aligned}$$

where R_{p,I_g} and R_{p,I_g^c} denote total revenues in the private sector by procurement firms and non-procurement firms respectively. Then, dividing by $\overline{\text{MRPK}}_g$ in both sides we obtain the expression for $\overline{\text{MRPK}}_p/\overline{\text{MRPK}}_g$

$$\frac{\overline{\text{MRPK}}_p}{\overline{\text{MRPK}}_g} = \left[\frac{R_{p,I_g}}{P_p Y_p} \left(\frac{\overline{\text{MRPK}}_{p,I_g}}{\overline{\text{MRPK}}_g} \right)^{-1} + \frac{R_{p,I_g^c}}{P_p Y_p} \left(\frac{\overline{\text{MRPK}}_{p,I_g^c}}{\overline{\text{MRPK}}_g} \right)^{-1} \right]^{-1} \tag{C.6}$$

Whenever $\overline{\text{MRPK}}_p \neq \overline{\text{MRPK}}_g$ there is misallocation of capital across sectors. The first term in equation (C.6) reflects the effects of within-firm misallocation on this between-sector misallocation. With $\sigma_g = \sigma_p$ this term would be equal to 1 if $\phi_g = \phi_p$ or if there were no financial frictions ($\lambda_i = 0 \forall i$). Instead, if $\phi_g > \phi_p$ firms switch their output relatively towards the public sector and hence $\overline{\text{MRPK}}_{p,I_g} > \overline{\text{MRPK}}_g$. The second term in equation (C.6) reflects both between-firm misallocation and selection into procurement.

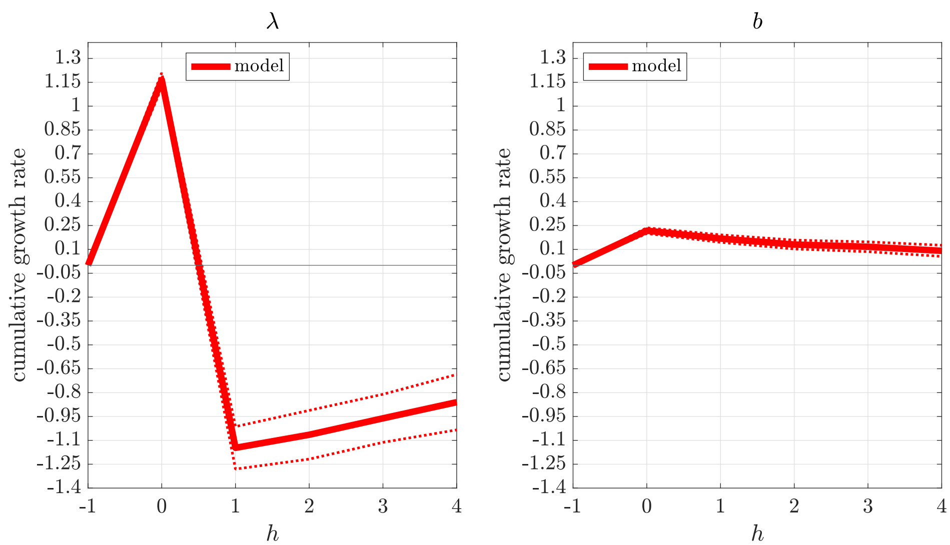
D Details on treatment effects in the benchmark calibration

To analyze dynamic treatment effects of procurement in our model and in the data, we estimate local projection panel regressions (Jordà, 2005). In particular, we regress the cumulative difference of a variable x, $\Delta_h \log(x_{i,t+h}) \equiv \log(x_{i,t+h}) - \log(x_{i,t-1})$ on the regressor PROC_{it}, the firm's lagged credit at $t-1$, firm fixed effects, and sector×year fixed effects:

$$\Delta_h \log(x_{i,t+h}) = \alpha_{ih} + \alpha_{sth} + \beta_1^h \text{PROC}_{it} + \beta_2^h \log\ x_{it-1} + \varepsilon_{ith+h} \tag{D.1}$$

where $x_{i,t+h}$ is either sales to the private sector, λ, or b, measured for firm i at time $t+h$. Therefore, $h = 0, 1, ..., H$ denotes the horizon at which the impact of procurement is estimated. We show the results from running these regressions at annual frequency.

Figure A.III. Treatment effects of procurement in the baseline calibration



Notes: This figure shows the cumulative estimated impact (β_1^h) of obtaining a procurement contract for different time horizons $h = 0, 1, 2, 3, 4$. The left panel shows the effects on λ. The right panel shows the effects on b.

References

Almunia, Miguel, David López-Rodríguez, and Enrique Moral-Benito, "Evaluating the Macro-Representativeness of a Firm-Level Database: An Application for the Spanish Economy," 2018. Banco de España Documento Ocasional 1802.

García-Santana, Manuel, Enrique Moral-Benito Josep Pijoan-Mas, and Roberto Ramos, "Growing like Spain," *International Economic Review*, 2020, *61* (1), 383–416.

Ivashina, Victoria, Luc Laeven, and Enrique Moral-Benito, "Loan types and the bank lending channel," *Journal of Monetary Economics*, Forthcoming.

Jordà, Oscar, "Estimation and Inference of Impulse Responses by Local Projections," *American Economic Review*, 2005, *95* (1), 161–182.